Small time asymptotics of the entropy of the heat kernel on a Riemannian manifold

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Joint work with Vlado Menkovski and Jim Portegies
(https://arxiv.org/abs/2209.11509)

Eindhoven University of Technology

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VAE with latent space $(\mathbb{R}^d, \mathcal{N}(0, \mathbb{I}_d))$

$(x_i) \subseteq \mathbb{R}^n$

Loss function: Evidence Lower Bound (ELBO)

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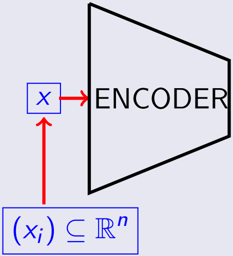

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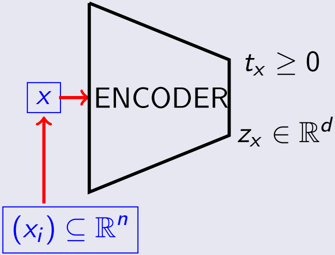

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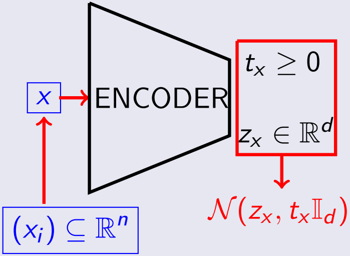

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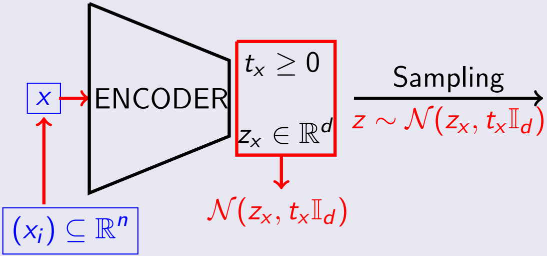

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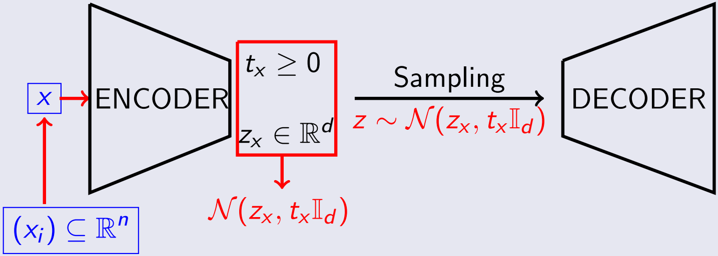

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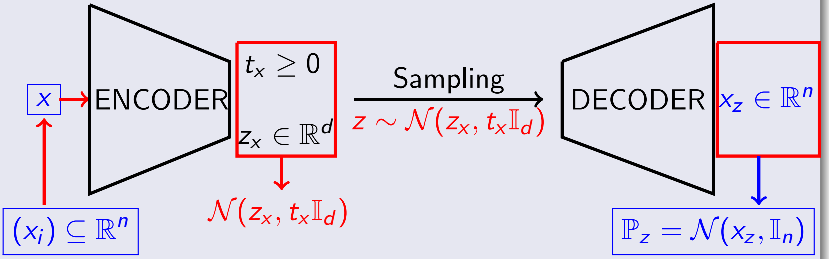

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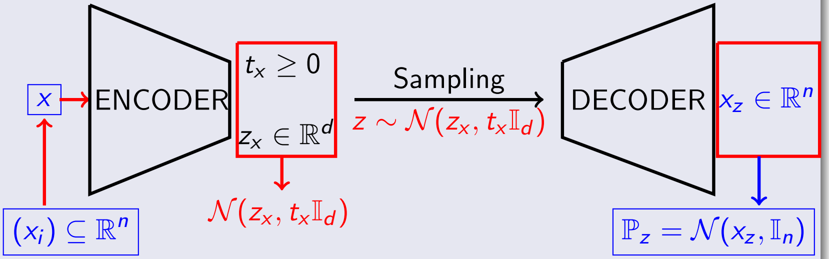

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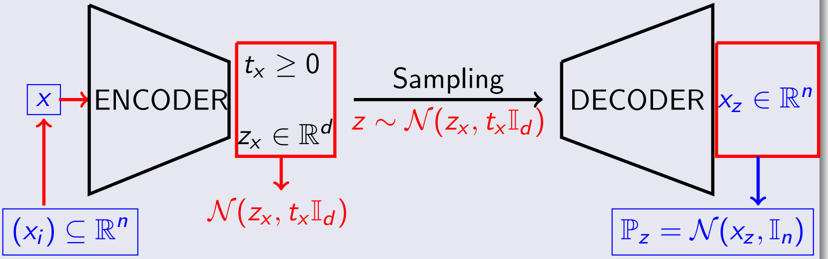

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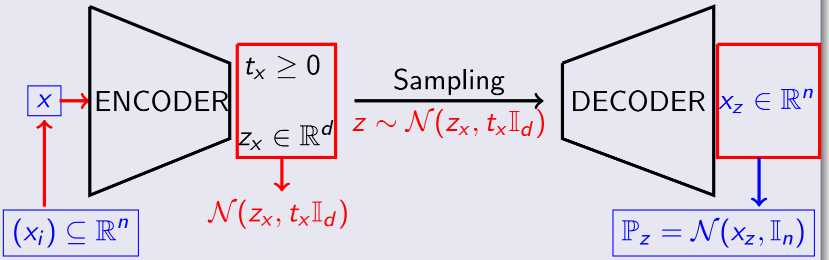

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$\exists\ c = c(z, \epsilon) \geq 0 :\ 0 \leq q_Z(t, z, w) \leq e^{-c/t}$ if $w \in Z \setminus B_\epsilon(z)$

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$\exists\, c = c(z,\epsilon) \geq 0 :\ 0 \leq q_Z(t,z,w) \leq e^{-c/t}$ if $w \in Z \setminus B_\epsilon(z)$

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$\implies$ Reduce the domain of integration

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Parametrix $u_i(0, y)$: in normal coordinates around z Rosenberg [4]

- $u_i(0, \cdot) : y \in \mathbb{R}^d \longmapsto u_i(0, y) \in \mathbb{R}$ has Taylor series at $y = 0$.

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Replace heat kernel with parametrix inside the Logarithm

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The first two terms

$$H[q_Z(t,z,\cdot)] = -\frac{d}{2}\log(2\pi t)\int_{B_\epsilon(0)} q_Z(t,0,y)\sqrt{\det[g(y)]}dy - \int_{B_\epsilon(0)} q_Z(t,0,y)\sqrt{\det[g(y)]}\frac{||y||^2}{2t}dy + \cdots$$

The first term $K_1(z,t)$: use exponential decay of the heat kernel

$$\begin{aligned} K_1(z,t) &= -\frac{d}{2}\log(2\pi t)\int_{B_\epsilon(0)} q_Z(t,0,y)\sqrt{\det[g(y)]}dy \\ &= -\frac{d}{2}\log(2\pi t)\int_{B_\epsilon(z)} q_Z(t,z,w)d\mathrm{Vol}(w) \\ &= -\frac{d}{2}\log(2\pi t)\int_{Z} q_Z(t,z,w)d\mathrm{Vol}(w) + o(t^n) \\ &= -\frac{d}{2}\log(2\pi t) + o(t^n). \end{aligned}$$

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$K_2(z,t)$: use Taylor expansion of the parametrix ...

$$K_2(z,t) = -\int_{B_\epsilon(0)} \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\left(-\frac{||y||^2}{2t}\right) \sum_{i=0}^{n} F^{(i)}_{(0,0)}(t,y)$$
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where

$$F : (t,y) \in \mathbb{R} \times \mathbb{R}^d \longmapsto \sqrt{\det[g(y)]} \sum_{i=0}^{n} u_i(0,y) t^i$$

[F is $\mathcal{C}^\infty$ when defined]

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Derivatives are multilinear forms

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Lemma ...

If $q = 2k + 1$ is odd, then

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$$= \sum_{p,k} t^{p+k} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} e^{\left(-\frac{||Y||^2}{2}\right)} P_{2k}(Y) dY + o(t^n)$$

Last equality follows by using properties of Gaussian integrals.

Conclusion

- We showed that

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for small values of t.

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Additional result: unit sphere $\mathcal{S}^d \subseteq \mathbb{R}^{d+1}$ $(d \geq 3)$

$$\frac{d}{2}log(2\pi t) - \frac{d}{2} + \frac{d(d-1)}{4}\, t + \frac{-15d^3+30d^2+5d-20}{720}\, t^2 + o(t^2).$$

References

- Rey, Luis A Pérez and Menkovski, Vlado and Portegies, Jacobus W
  Diffusion variational autoencoders.
- Minakshisundaram, Subbaramiah and Pleijel, Åke
  Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds.
- Kingma, Diederik P and Welling, Max
  Auto-encoding variational bayes.
- Rosenberg, Steven
  The Laplacian on a Riemannian manifold: an introduction to analysis on manifolds.
- Sakai, Takashi
  On eigen-values of Laplacian and curvature of Riemannian manifold.