Regularization Properties of Dropout

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Example: Student-teacher setup [1]

Teacher: a small (e.g. $m = 3$) shallow ReLU neural network:

$$f_t(\tilde{w}, x) = \frac{1}{m} \sum_{i=1}^{m} (\tilde{w}_i^T x)_+.$$

Student: a wide ($n \gg m$) shallow ReLU neural network:

$$f(w, x) = \frac{\beta(n)}{n} \sum_{i=1}^{n} (w_i^T x)_+.$$

Goal: train student network to reconstruct the teacher network

$$\min_w \mathcal{L}(w)$$

where

$$\mathcal{L}(w) = \|f_t(\tilde{w}, x) - f(w, x)\|^2_{L_2(\mu_X)}$$

Example: Student-teacher setup [1]

Let $x, w_i \in \mathbb{R}^2$ and use (sub)gradient descent to find a (local) minima

$$w(k+1) = w(k) - \alpha \nabla_w \mathcal{L}(w(k))$$

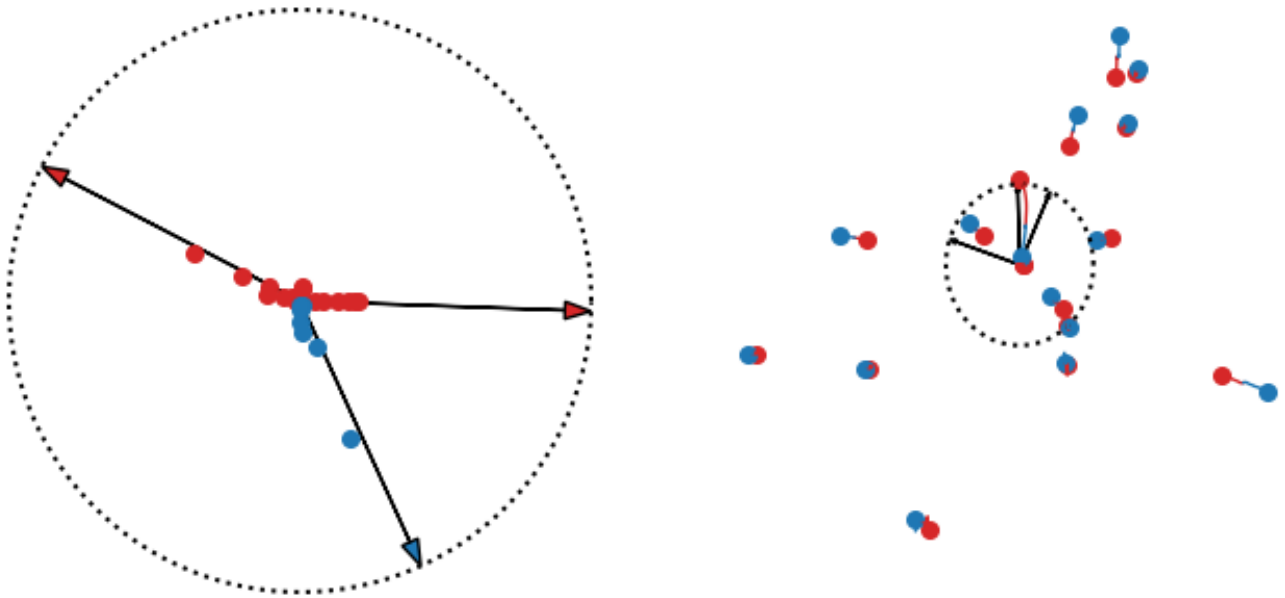

Figure: Dots are weights $\{w_i\}$ of the student network after convergence for $\beta(n) = 1$ (left) and $\beta(n) = \sqrt{n}$ (right). Arrows are the weights of the teacher network $\{\tilde{w}_i\}$

Bernoulli Dropout [2, 3]

Consider a shallow neural network with dropout:

$$f_d(w, \eta, x) = \frac{\beta(n)}{n} \sum_{i=1}^{n} (1 + \eta_i)(w_i^T x)_+,$$

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and the loss:

$$\mathcal{L}_d(w, \eta) = \|f_t(\tilde{w}, x) - f_d(w, \eta, x)\|^2_{L_2(\mu_X)}$$

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where

$$\eta_i(k) = \begin{cases} -1, \text{ w.p. } p \\ \frac{p}{1-p} \text{ w.p. } 1-p \end{cases}$$

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Note:

$$\mathbb{E}_\eta f_d(w, \eta, x) = f(w, x)$$

Bernoulli Dropout: Regularization Effect

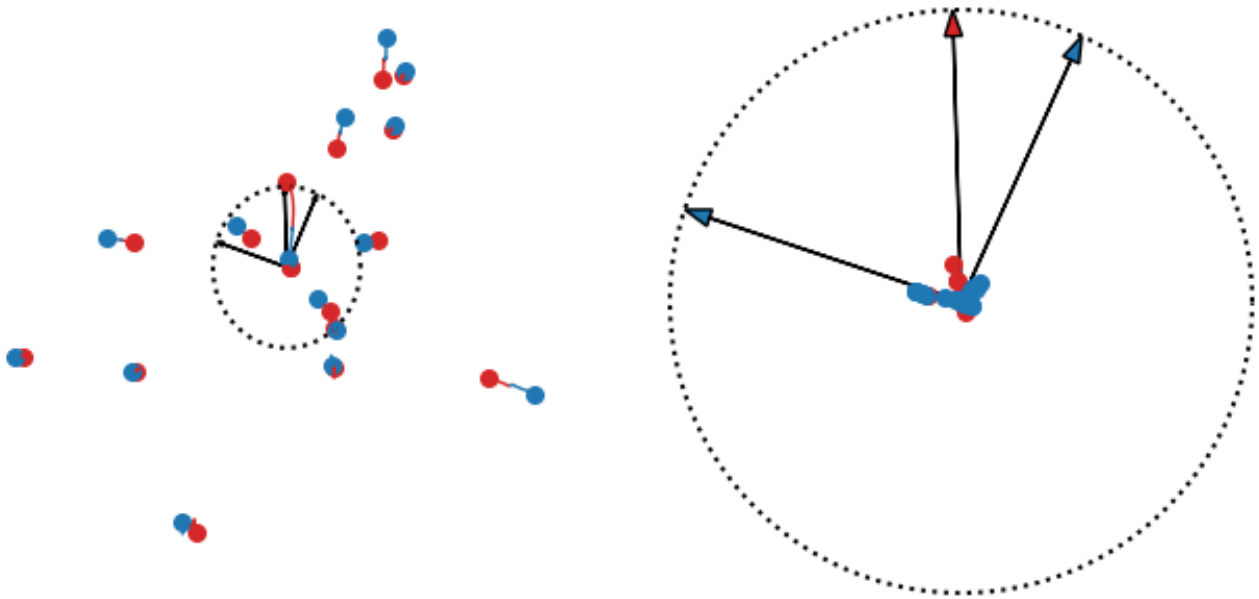

Figure: Dots are weights $\{w_i\}$ of the student network after convergence for $\beta(n) = \sqrt{n}$ without dropout (left) and with dropout (right). Arrows are the weights of the teacher network $\{\tilde{w}_i\}$

Ornstein–Uhlenbeck Dropout

For the same shallow neural network

$$f_d(w, \eta, x) = \frac{\beta(n)}{n} \sum_{i=1}^{n} (1 + \eta_i)(w_i^T x)_+,$$

$$\mathcal{L}_d(w, \eta) = \|f_t(\tilde{w}, x) - f_d(w, \eta, x)\|^2_{L_2(\mu_X)}$$

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we introduce a continuous analog of the dropout:

Bernoulli DO:

$$w^{k+1} = w^k - \alpha \nabla_w \mathcal{L}_d(w^k, \eta^k)$$

Ornstein–Uhlenbeck DO:

$$\dot{w} = -\alpha \nabla_w \mathcal{L}_d(w, \eta_t)$$

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$$\eta_i^k = \begin{cases} -1, & \text{w.p. } p \\ \frac{p}{1-p}, & \text{w.p. } 1-p \end{cases}$$

$$w^{k+1} = w^k - \alpha \nabla_w \mathcal{L}_d(w^k, \eta^k)$$

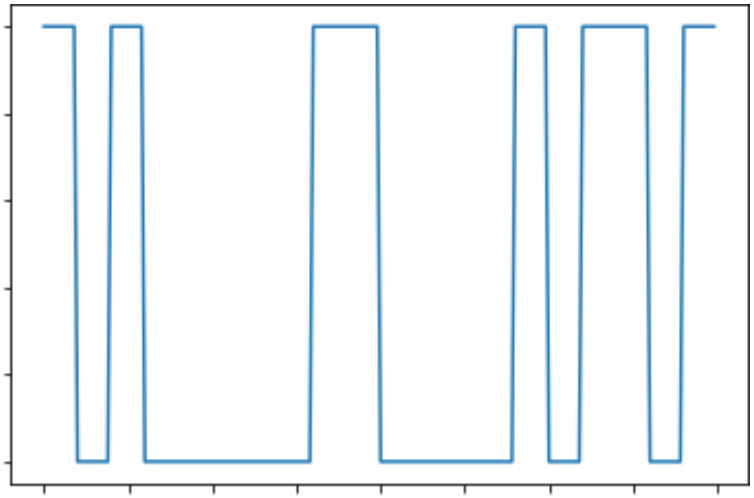
Ornstein–Uhlenbeck DO:

$$d\eta_t = -\gamma \eta_t dt + \sqrt{2\gamma\sigma^2} db_t$$

$$\dot{w} = -\alpha \nabla_w \mathcal{L}_d(w, \eta_t)$$

Ornstein–Uhlenbeck Dropout

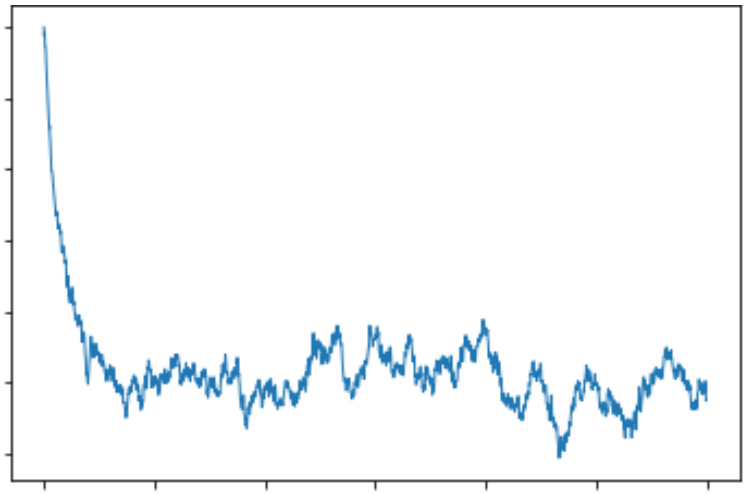
Bernoulli DO:



$$\mathbb{E}\eta^k = 0$$

$$cov(\eta^k, \eta^l) = \frac{p}{p-1}\delta_{kl}$$

Ornstein–Uhlenbeck DO:



$$\mathbb{E}\eta_t = \eta_0 e^{-\gamma t}$$

$$cov(\eta_t, \eta_s) = \sigma^2 e^{-\gamma|t-s|}$$

Our Approach (a teaser)

- We assume that minimizers of $\mathcal{L}(w)$ form a nice manifold
- We use Katzenberger's results [4] to study the training dynamic in the small noise limit $\sigma \to 0$
- We focus on a special case of uncorrelated noise $\lim_{\gamma\to\infty} \lim_{\sigma\to 0}$

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